Bounds on the Extension of Antennas for Stable Spinning Satellites

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Introduction

LET us consider a force-free satellite spinning about the axis of maximum moment of inertia. If the satellite is rigid, then from rigid-body dynamics we conclude that the rotational motion is stable.[1] Next, let us assume that a pair of thin flexible rods with given tip masses are being deployed slowly, so that they are extended at equal rates but in opposite directions along the spin axis. The question is being asked as to how far can the rods be extended without causing the satellite motion to become unstable. It is assumed that the rate of extension of the rods is sufficiently small that any Coriolis effects that might arise because of the rate of change in length can be ignored. It is also assumed that the mode of deformation of the flexible rods is antisymmetric, with the implication that the satellite mass center does not shift relative to the rigid part. The satellite is shown in Fig. 1.

In an early attempt to investigate the flexibility effects on the stability of a force-free spinning satellite, Meirovitch and Nelson[2] considered two mathematical models related to that used here. In fact, the two mathematical models used in Ref. 2 can be obtained as limiting cases of that used here by assuming in one case that the rods act like massless springs and in the other case that there are no tip masses. Stability was investigated in Ref. 2 by means of an infinitesimal analysis. More recently, Meirovitch and Calico[3] have formulated the general problem of stability of motion of a force-free spinning satellite with flexible appendages, with stability being investigated via Liapunov's second method. For stability, the system Hamiltonian must be positive definite. Moreover, because of internal energy dissipation, however slight, the equilibrium is asymptotically stable if the Hamiltonian is positive definite in the neighborhood of the equilibrium point in question. Of particular interest in Ref. 3 is the use of the "method of integral coordinates," whereby the spatial dependence, introduced by the variables describing the deformation of the continuous elastic members, is eliminated from the Hamiltonian by the use of Schwartz's inequality for functions. The present work is based on the formulation of Ref. 3.

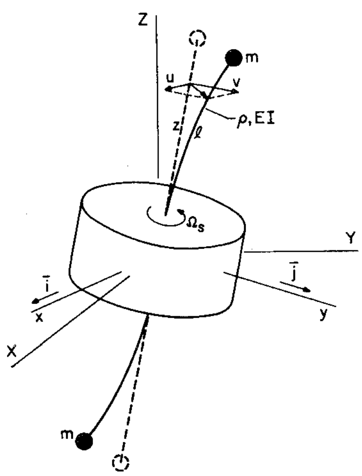

Fig. 1 The spinning satellite with flexible antennas.

Received September 28, 1973; revision received December 7, 1973. The author wishes to thank Drs. J. V. Fedor and T. W. Flatley of Stabilization and Control Branch, NASA Goddard Space Flight Center, for suggesting the problem.

Index categories: Spacecraft Attitude Dynamics and Control; Structural Dynamic Analysis.

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Derivation of the Stability Criteria

From Ref. 3, we conclude that the kinetic energy has the form

$$T = \tfrac{1}{2}\{\omega\}^T[J]\{\omega\} + \{K\}^T\{\omega\} + T_{EL} \tag{1}$$

where $\{\omega\}$ is the angular velocity vector, and

$$[J] = [J]_0 + [J]_1 \tag{2}$$

is the inertia matrix, in which $[J]_0$ is the inertia matrix of the entire body in underformed state, having the elements

$$\begin{gathered} J_{011} = A, \quad J_{022} = B, \quad J_{033} = C \\ J_{012} = J_{021} = J_{013} = J_{031} = J_{023} = J_{032} = 0 \end{gathered} \tag{3}$$

and $[J]_1$ is the change in the inertia matrix due to elastic deformations. For antisymmetric displacements, $u(z, t) = u(-z, t)$, $v(z, t) = v(-z, t)$, the elements of $[J]_1$ are

$$\begin{aligned} J_{111} &= 2\int_h^{h+l} \rho v^2\,dz + 2mv^2(h+l, t) \\ J_{122} &= 2\int_h^{h+l} \rho u^2\,dz + 2mu^2(h+l, t) \\ J_{133} &= 2\int_h^{h+l} \rho(u^2+v^2)\,dz + 2m[u^2(h+l, t)+v^2(h+l, t)] \\ J_{112} &= J_{121} = -2\int_h^{h+l} \rho uv\,dz - 2mu(h+l, t)v(h+l, t) \\ J_{113} &= J_{131} = -2\int_h^{h+l} \rho zu\,dz - 2m(h+l)u(h+l, t) \\ J_{123} &= J_{132} = -2\int_h^{h+l} \rho zv\,dz - 2m(h+l)v(h+l, t) \end{aligned} \tag{4}$$

In addition, $\{K\}$ can be identified as the angular momentum matrix due to elastic velocities alone, and its elements are

$$\begin{aligned} K_1 &= -2\int_h^{h+l} \rho z\dot{v}\,dz - 2m(h+l)\dot{v}(h+l, t) \\ K_2 &= 2\int_h^{h+l} \rho z\dot{u}\,dz + 2m(h+l)\dot{u}(h+l, t) \\ K_3 &= 2\int_h^{h+l} \rho(u\dot{v}-v\dot{u})\,dz + 2m[u(h+l, t)\dot{v}(h+l, t) - v(h+l, t)\dot{u}(h+l, t)] \end{aligned} \tag{5}$$

whereas T_{EL} represents the kinetic energy due to elastic velocities alone, namely,

$$T_{EL} = \int_h^{h+l} \rho(\dot{u}^2+\dot{v}^2)\,dz + m[\dot{u}^2(h+l, t)+\dot{v}^2(h+l, t)] \tag{6}$$

In a force-free environment, the only potential energy is elastic. Hence, because the rods undergo flexure alone, the potential energy is

$$V_{EL} = \int_h^{h+l} EI\left[\left(\frac{\partial^2 u}{\partial z^2}\right)^2 + \left(\frac{\partial^2 v}{\partial z^2}\right)^2\right]dz \tag{7}$$

But in the absence of external torques the angular momentum is constant in magnitude and direction. Hence, let us write

$$\{L\} = \left\{\frac{\partial T}{\partial \omega}\right\} = [J]\{\omega\} + \{K\} = \{\beta\} \tag{8}$$

where $\{\beta\}$ is the matrix of the conserved angular momentum. Equation (8) can be solved for $\{\omega\}$, with the result

$$\{\omega\} = [J]^{-1}\{\beta - K\} \tag{9}$$

Introducing Eq. (9) into (1), we can write the kinetic energy in the form

$$T = T_2 + T_0 \tag{10}$$

where

$$T_2 = T_{EL} - \tfrac{1}{2}\{K\}^T[J]^{-1}\{K\} \tag{11}$$

is quadratic in the elastic velocities, and

$$T_0 = \tfrac{1}{2}\{\beta\}^T[J]^{-1}\{\beta\} \tag{12}$$

contains no velocities at all.

The matrix $\{\beta\}$ depends only on the angular coordinates. Moreover, it depends only on two of the three angular coordinates. If the body spins initially undeformed with the constant angular velocity Ω_s about the inertial axis Z, so that the magnitude of the angular momentum is $\beta = C\Omega_s$, then after some perturbation the angular momentum can be written as $\{\beta\} = \beta\{l\} = C\Omega_s\{l\}$, where $\{l\}$ is the matrix of the direction cosines between Z and the body axes 1, 2, 3. If the body axes are obtained from the inertial system X, Y, Z by the rotations θ_3 about 3 (where 3 coincides initially with Z), θ_1 about 1, and θ_2 about 2, then

$$\{\beta\} = C\Omega_s\{l\} = C\Omega_s \begin{Bmatrix} -\cos\theta_1 \sin\theta_2 \\ \sin\theta_1 \\ \cos\theta_1 \cos\theta_2 \end{Bmatrix} \tag{13}$$

so that $\{\beta\}$ depends only on θ_1 and θ_2.

But the elastic displacements are such that (Ref. 3)

$$V_{EL} \geqq V_{EL}^* = \Lambda_1^2 \left\{ \int_h^{h+l} \rho(u^2+v^2)\,dz + m[u^2(h+l,t)+v^2(h+l,t)] \right\} \tag{14}$$

where Λ_1 is the first natural frequency associated with the vibration u or v. Moreover, T_2 is positive definite by definition. It follows that the Hamiltonian is positive definite, hence, the spinning motion is asymptotically stable, if the functional

$$\kappa = T_0 + V_{EL}^* \tag{15}$$

is positive definite (Ref. 3).

We shall consider the case in which the angular displacements and the elastic deformations are small. Recalling Eq. (2), where $[J]_1$ is small compared to $[J]_0$, it can be shown that

$$[J]^{-1} \cong [J]_0^{-1} - [J]_0^{-1}[J]_1[J]_0^{-1} + [J]_0^{-1}[J]_1[J]_0^{-1}[J]_1[J]_0^{-1} \tag{16}$$

so that, inserting Eqs. (12), (13), (14), and (16) into Eq. (15), and letting θ_1, θ_2, u, and v be small, we obtain

$$\begin{aligned}\kappa_1 = \frac{1}{2}\Omega_s^2 \Bigg\{ & \frac{C}{B}(C-B)\theta_1^2 + \frac{C}{A}(C-A)\theta_2^2 - \\ & 4\frac{C}{A}\left[\int_h^{h+l} \rho z u\,dz + m(h+l)u(h+l,t)\right]\theta_2 + \\ & 4\frac{C}{B}\left[\int_h^{h+l} \rho z v\,dz + m(h+l)v(h+l,t)\right]\theta_1 - \\ & 2\left[\int_h^{h+l} \rho(u^2+v^2)\,dz + mu^2(h+l,t) + mv^2(h+l,t)\right] + \\ & \frac{4}{A}\left[\int_h^{h+l} \rho z u\,dz + m(h+l)u(h+l,t)\right]^2 + \\ & \frac{4}{B}\left[\int_h^{h+l} \rho z v\,dz + m(h+l)v(h+l,t)\right]^2 \Bigg\} + \\ & \Lambda_1^2 \left\{ \int_h^{h+l} \rho(u^2+v^2)\,dz + m[u^2(h+l,t)+v^2(h+l,t)] \right\}\end{aligned} \tag{17}$$

Next let us introduce the integral coordinates

$$\begin{aligned}\bar{u}(t) &= \int_h^{h+l} \rho z u(z,t)\,dz + m(h+l)u(h+l,t) \\ \bar{v}(t) &= \int_h^{h+l} \rho z v(z,t)\,dz + m(h+l)v(h+l,t)\end{aligned} \tag{18}$$

and consider the following Schwartz's inequalities

$$\begin{aligned}\left[\int_h^{h+l} \rho z u\,dz + m(h+l)u(h+l,t)\right]^2 &\leqq \left[\int_h^{h+l} \rho z^2\,dz + m(h+l)^2\right] \times \left[\int_h^{h+l} \rho u^2\,dz + mu^2(h+l,t)\right] \\ \left[\int_h^{h+l} \rho z v\,dz + m(h+l)v(h+l,t)\right]^2 &\leqq \left[\int_h^{h+l} \rho z^2\,dz + m(h+l)^2\right] \times \left[\int_h^{h+l} \rho v^2\,dz + mv^2(h+l,t)\right]\end{aligned} \tag{19}$$

Moreover, we recognize that

$$\int_h^{h+l} \rho z^2\,dz + m(h+l)^2 = \tfrac{1}{2}A_1 = \tfrac{1}{2}B_1 \tag{20}$$

represents the moment of inertia of the rods and tip masses in underformed state about a transverse axis, so that we can write $A = A_0 + A_1$, $B = B_0 + B_1$, where A_0 and B_0 are the moments of inertia of the rigid part alone about axes 1 and 2, respectively. Then, considering definitions (18), it follows that

$$\int_h^{h+l} \rho u^2\,dz + mu^2(h+l,t) \geqq \frac{2\bar{u}^2(t)}{A_1}, \qquad \int_h^{h+l} \rho v^2\,dz + mv^2(h+l,t) \geqq \frac{2\bar{v}^2(t)}{B_1} \tag{21}$$

In view of the above, we can write

$$\begin{aligned}\kappa_1 \geqq \tfrac{1}{2}\Omega_s^2 \Bigg[& \frac{C}{B}(C-B)\theta_1^2 + \frac{C}{A}(C-A)\theta_2^2 - 4\frac{C}{A}\theta_2\bar{u} + \\ & 4\frac{C}{B}\theta_1\bar{v} + \frac{4}{A}\bar{u}^2 + \frac{4}{B}\bar{v}^2\Bigg] + 2(\Lambda_1^2 - \Omega_s^2)\left(\frac{1}{A_1}\bar{u}^2 + \frac{1}{B_1}\bar{v}^2\right) = \kappa_2\end{aligned} \tag{22}$$

from which we conclude that if κ_2 is positive definite, then κ_1 is positive definite, and the equilibrium is asymptotically stable. Note that κ_2 is free of the spatial variable z. Introducing the notation

$$\frac{\Omega_s}{\Lambda_1} R_\Omega, \qquad \frac{A_1}{A_0} = R_A, \qquad \frac{B_1}{B_0} = R_B \tag{23}$$

where the first is referred to as the spin ratio and the latter two as moments of inertia ratios, it can be shown that the Hessian matrix associated with κ_2 is

$$[\mathscr{H}] = \tfrac{1}{2}\Omega_s^2 \times \begin{bmatrix} \frac{C}{B}(C-B) & 0 & 0 & 2\frac{C}{B} \\ 0 & \frac{C}{A}(C-A) & -2\frac{C}{A} & 0 \\ 0 & -2\frac{C}{A} & 4\left(\frac{1}{A} + \frac{1}{A_0}\frac{1-R_\Omega^2}{R_A R_\Omega^2}\right) & 0 \\ 2\frac{C}{B} & 0 & 0 & 4\left(\frac{1}{B} + \frac{1}{B_0}\frac{1-R_\Omega^2}{R_B R_\Omega^2}\right) \end{bmatrix} \tag{24}$$

According to Sylvester's criterion, $[\mathscr{H}]$ is positive definite if all its principal minor determinants are positive.[1] The conditions for $[\mathscr{H}]$ to be positive definite can be shown to reduce to

$$C > A \tag{25a}$$

$$C > B \tag{25b}$$

$$R_\Omega < \left[1 + \frac{R_A}{(C/A_0) - 1 - R_A}\right]^{-1/2} \tag{25c}$$

$$R_\Omega < \left[1 + \frac{R_B}{(C/B_0) - 1 - R_B}\right]^{-1/2} \tag{25d}$$

Hence, the equilibrium is asymptotically stable if the system parameters are such that inequalities (25) are satisfied. Conditions

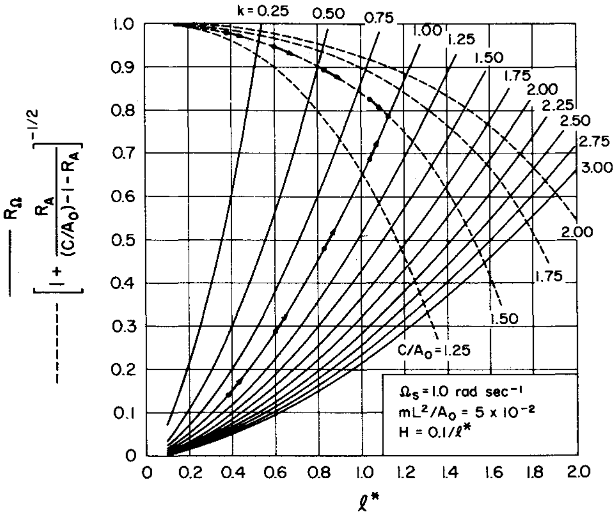

Fig. 2 Stability diagram in the parameter plane.

(25a) and (25b) are recognized as the requirement that the spin axis be that of "the greatest moment of inertia." On the other hand, the desired conditions, namely, those yielding the maximum length to which the rods can be extended without impairing stability, can be obtained from inequalities (25c) and (25d).

Numerical Results

Inequalities (25c) and (25d) can be used to derive stability diagrams in terms of the system parameters. Because information that can be extracted from inequality (25d) is analogous to that obtained from inequality (25c), we shall be concerned only with the latter.

Let us assume for simplicity that the rods are uniform, ρ = const, EI = const, and introduce the dimensionless quantities $\rho l/m = l^*$ and $h/l = H$, as well as the reference length $L = m/\rho$, which enables us to write the inertia ratio in the form

$$R_A = \frac{2}{A_0}\left[\int_h^{h+l} \rho z^2\,dz + m(h+l)^2\right] = \frac{2mL^2}{A_0}\, l^{*2}[1+\tfrac{1}{3}l^*(1+3H+3H^2)] \qquad (26)$$

The first natural frequency Λ_1 is obtained from the solution of the eigenvalue problem associated with a uniform cantilever bar with a tip mass. It can be shown to have the value

$$\Lambda_1 = k\frac{(\alpha_1 l)^2}{l^{*2}}, \qquad k = (EI/\rho L^4)^{1/2} \qquad (27)$$

where $\alpha_1 l$ is the lowest solution of the characteristic equation

$$(1+\cos\alpha l\cosh\alpha l) = \alpha l\frac{1}{l^*}(\sin\alpha l\cosh\alpha l - \cos\alpha l\sinh\alpha l) \quad (28)$$

Corresponding to a given l^*, Eq. (28) can be solved for $\alpha_1 l$. Then for a given value of k, Eq. (27) yields Λ_1, which can be used to calculate $R_\Omega = \Omega_s/\Lambda_1$. Figure 2 shows in solid lines plots R_Ω vs l^* for $\Omega_s = 1.0$ rad/sec, with k playing the role of a parameter, and in dashed lines plots $\{1+R_A/[(C/A_0)-1-R_A]\}^{-1/2}$ vs l^* for $mL^2/A_0 = 5\times10^{-2}$ and $H = 0.1/l^*$, with C/A_0 playing the role of a parameter. Similar diagrams can be obtained for various other combinations of Ω_s, mL^2/A_0, and H. The system is stable for values of l^* for which the curve R_Ω vs l^* remains below the curve $\{1+R_A/[(C/A_0)-1-R_A]\}^{-1/2}$ vs l^*. As the antennas are being deployed slowly, hence, as l^* increases, R_Ω and $\{1+R_A/[(C/A_0)-1-R_A]\}^{-1/2}$ follow the corresponding curves, as indicated by arrows in Fig. 2. At the intersection of the two curves we have $R_\Omega = \{1+R_A/[(C/A_0)-1-R_A]\}^{-1/2}$, and beyond that point $R_\Omega > \{1+R_A/[(C/A_0)-1-R_A]\}^{-1/2}$. It follows that stability can be expected as long as l^* stays below the value corresponding to the intersection of the two curves.

As an illustration, let us consider:

$$m = 5 \text{ slug}, \quad \rho = 0.1 \text{ slug/in.}, \quad EI = 31.25\times10^6 \text{ lb in.}^2,$$
$$h = 5 \text{ in.}, \quad A_0 = 25\times10^6 \text{ slug in.}^2, \quad C = 37.5\times10^6 \text{ slug in.}^2,$$
$$\Omega_s = 1.0 \text{ rad/sec}$$

Using the above data, we can calculate the following:

$$L = m/\rho = 50 \text{ in.}, \quad k = (EI/\rho L^4)^{1/2} = 1.0, \quad C/A_0 = 1.5$$
$$mL^2/A_0 = 5\times10^{-2}, \quad H = h/Ll^* = 0.1/l^*$$

so that Fig. 2 is applicable. (This is no coincidence, as the data was purposely chosen to render Fig. 2 applicable.) The curves corresponding to $k = 1.0$ and $C/A_0 = 1.5$ intersect at $l^* = 1.13$. Hence, the motion is stable as long as $l = l^*L < 56.5$ in.

Conclusions

A method is presented whereby it is possible to determine how far a pair of antennas can be deployed without destabilizing a spinning satellite. The analysis is based on Liapunov's direct method in conjunction with the method of the integral coordinates. The numerical results are presented in the form of a parameter plot that can be used for design purposes.

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Optimal Guidance with Maneuvering Targets

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Introduction

In the derivation of optimal guidance laws for nonlinear missile systems several inherent assumptions are made in order to allow a feedback law to be obtained. Two of the more common simplifying assumptions made are that the missile has infinite bandwidth and that the target is assumed nonmaneuvering. Cottrell,[1,2] considers an optimal derivation based upon the assumption that the missile has finite bandwidth, i.e., a first-order lag model. His optimal derivation, however, did not consider a maneuvering target but rather adds a term in the guidance law based upon intercept kinematics.

The present Note considers the development of the optimal guidance law for a missile system with finite bandwidth and with a maneuvering target. It is shown that the results reduce to

Received January 18, 1973; revision received November 29, 1973.

Index categories: Navigation, Control, and Guidance Theory.

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